7 Control Charts for Attributes

- Quality characteristics that can be classified as conforming or nonconforming are called **attributes**. Four widely-used attributes control charts are:
    1. p chart: fraction of nonconforming units
    2. np chart: number of nonconforming units
    3. c chart: total number of nonconformities per unit
    4. u chart: average number of nonconformities per unit

7.1 The *p*-chart and *np*-chart with Equal Sample Sizes

7.1.1 *p*-chart for Known or Specified *p*

- Situation: each production unit is classified as either conforming or nonconforming.
- The parameter of interest is p, the **fraction nonconforming**. The fraction nonconforming is the ratio of the number of nonconforming items in a population to the total number of items in a population.
- Assumptions:
    - Each unit is assumed to have an equal probability of being judged nonconforming. Therefore, the inspection of one single unit can be viewed as a Bernoulli experiment.
    - If a sample of n independent units is selected from the same production process and inspected, this then becomes a binomial experiment.
    - Repeatedly take samples of size n. Let D_i be the number of nonconforming units and $\widehat{p}_i = D_i/n$ the proportion nonconforming in the i^{th} sample.
    - Each D_i follows a binomial distribution with parameters n and p.
- By the Central Limit Theorem, for large n, each $\widehat{p}_i$ is approximately normally distributed with $\mu_{\widehat{p}} = p$ and $\sigma_{\widehat{p}} = \sqrt{\frac{p(1-p)}{n}}$. That is, $\widehat{p}_i \sim$
- The **control limits for *p*-chart** are:

$$\begin{aligned} \text{UCL} &= p + 3 \\ \text{Centerline} &= p \\ \text{LCL} &= p - 3 \qquad . \end{aligned} \tag{12}$$

- If UCL > 1 then reset UCL=1. If LCL < 0 then reset LCL=0.
- As long as $\widehat{p}$ remains within control limits and no systematic pattern is evident, we conclude the process is *in control at level* p.
- If $\widehat{p}$ is outside the control limits or a systematic pattern is evident, we conclude the process has shifted to a new level and is *out of control at level* p.
- These formulas are based on a known value of p. This value may be specified by management or may be known from extensive research.

7.1.2 *np*-Chart for Known or Specified *p*

- As an alternative to the *p*-chart when n is constant, it is possible to use a control chart *based on the number of nonconforming units*. This is called an <u>*np*-chart</u>.
- The **control limits for *np*-chart** are:

$$\begin{aligned} \text{UCL} &= np + 3 \\ \text{Centerline} &= np \\ \text{LCL} &= np - 3 \qquad\qquad . \end{aligned} \tag{13}$$

EXAMPLE: Thirty sample batches of 500 electronic circuits are tested and the number of failing circuits is recorded. An in-control process is assumed to have 2% defective circuits. Make p and np charts. The numbers failing for each batch of 500 tested circuits are given in the SAS code.

SAS Code for *p*-chart (*p* known)

```
DM 'LOG; CLEAR; OUT; CLEAR;';
* ODS PRINTER PDF file='C:\COURSES\ST528\SAS\PCHART2.PDF';
ODS LISTING;
OPTIONS NODATE NONUMBER LS=76 PS=54;

*****************************************************;
***          P-chart (with specified p)           ***;
*** The response is the number of failing circuits ***;
***          from 30 batches of size 500          ***;
*****************************************************;
DATA circuits;
  INPUT batch fail @@;
LINES;
 1   5     2   6     3  11     4   6     5   4
 6   9     7  17     8  10     9  12    10   9
11   8    12   7    13   7    14  15    15   8
16  18    17  12    18  16    19   4    20   7
21  17    22  12    23   8    24   7    25  15
26   6    27   8    28  12    29   7    30   9
;
/* You can Specify the Standard Average Proportion with
   the P0= Option (as Shown in the Following Statments) */

TITLE1 'p Chart for Failing Circuits';
TITLE2 'Using Data in CIRCUITS and Standard Value P0=0.02';
SYMBOL1 V=star C=black;

PROC SHEWHART DATA=circuits ;
   PCHART fail*batch='1' / TESTS = 1 to 8   SUBGROUPN = 500
                       P0    = 0.02     PSYMBOL   = p0
                       ZONELABELS TABLETESTS TABLELEGEND NEEDLES;
   LABEL batch = 'Batch Number'
         fail  = 'Fraction Failing';
RUN;
```

SAS Code for *np*-chart (*p* known)

Replace PCHART and PSYMBOL=p0 with NPCHART and NPSYMBOL = np0.

SAS Output for p-chart and np-chart (p known)

p Chart for Failing Circuits
Using Data in CIRCUITS and Standard Value P0=0.02

The SHEWHART Procedure

p Chart Summary for fail

batch	Subgroup Sample Size	-3 Sigma Limits with n=500 for Proportion- Lower Limit	Subgroup Proportion	Upper Limit	Special Tests Signaled
1	500	0.00121703	0.01000000	0.03878297	
2	500	0.00121703	0.01200000	0.03878297	
3	500	0.00121703	0.02200000	0.03878297	
4	500	0.00121703	0.01200000	0.03878297	
5	500	0.00121703	0.00800000	0.03878297	6
6	500	0.00121703	0.01800000	0.03878297	
7	500	0.00121703	0.03400000	0.03878297	
8	500	0.00121703	0.02000000	0.03878297	
9	500	0.00121703	0.02400000	0.03878297	
10	500	0.00121703	0.01800000	0.03878297	
11	500	0.00121703	0.01600000	0.03878297	
12	500	0.00121703	0.01400000	0.03878297	
13	500	0.00121703	0.01400000	0.03878297	
14	500	0.00121703	0.03000000	0.03878297	
15	500	0.00121703	0.01600000	0.03878297	
16	500	0.00121703	0.03600000	0.03878297	
17	500	0.00121703	0.02400000	0.03878297	
:	:	:	:	:	
29	500	0.00121703	0.01400000	0.03878297	
30	500	0.00121703	0.01800000	0.03878297	

Test Descriptions
Test 6   Four out of five points in a row in Zone B or beyond

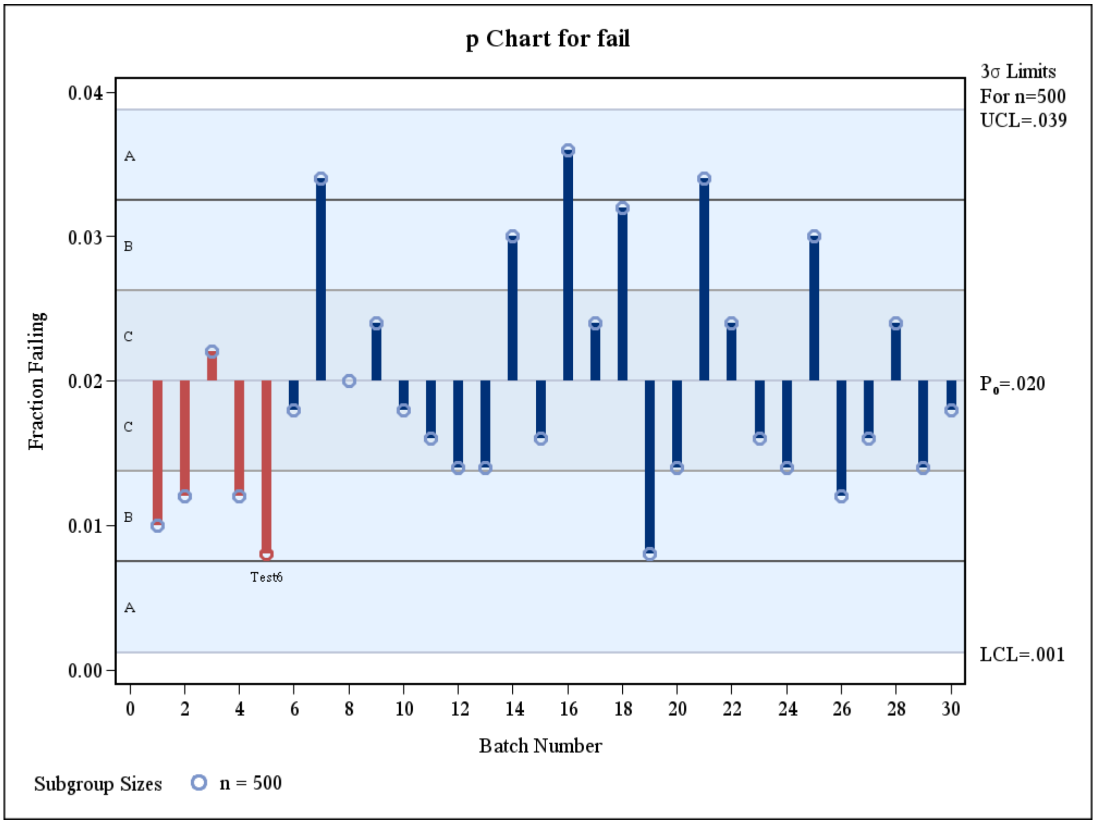
np Chart for Failing Circuits
Using Data in CIRCUITS and Standard Value P0=0.02

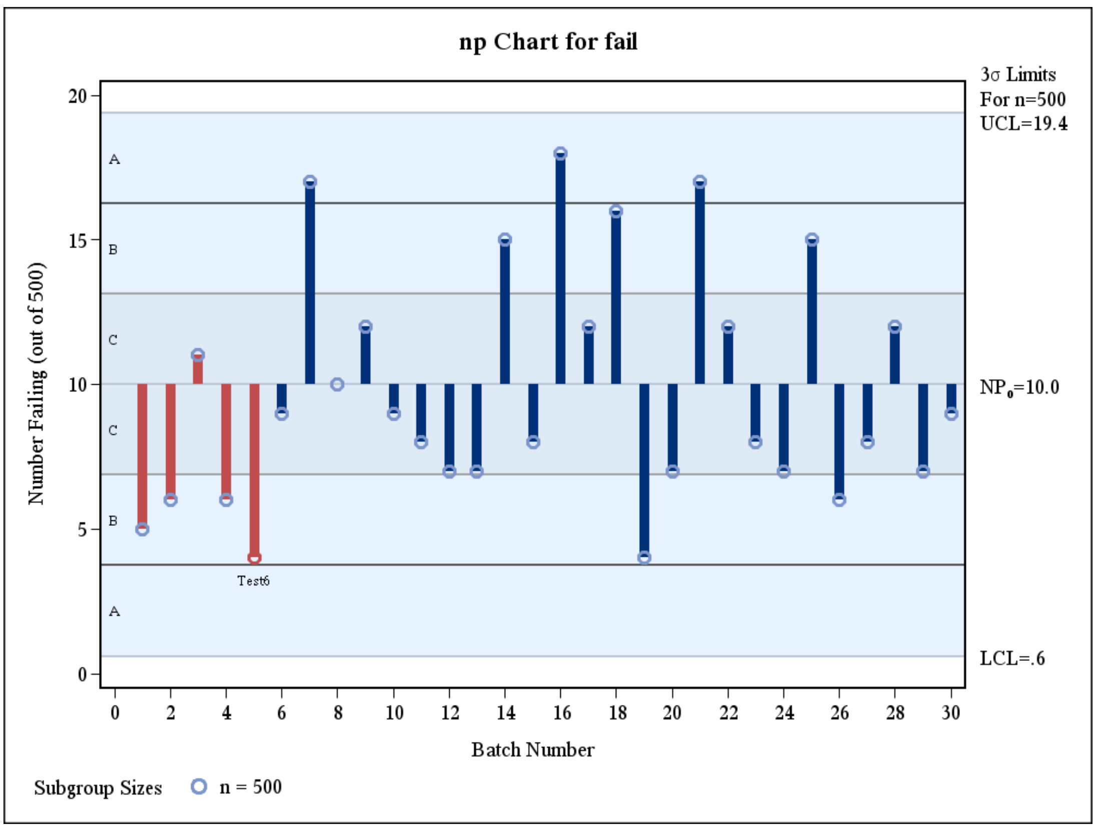
The SHEWHART Procedure

np Chart Summary for fail

batch	Subgroup Sample Size	-3 Sigma Limits with n=500 for Number- Lower Limit	Subgroup Number	Upper Limit	Special Tests Signaled
1	500	0.60851449	5.000000	19.391486	
2	500	0.60851449	6.000000	19.391486	
3	500	0.60851449	11.000000	19.391486	
4	500	0.60851449	6.000000	19.391486	
5	500	0.60851449	4.000000	19.391486	6
6	500	0.60851449	9.000000	19.391486	
7	500	0.60851449	17.000000	19.391486	
8	500	0.60851449	10.000000	19.391486	
9	500	0.60851449	12.000000	19.391486	
10	500	0.60851449	9.000000	19.391486	
11	500	0.60851449	8.000000	19.391486	
12	500	0.60851449	7.000000	19.391486	
13	500	0.60851449	7.000000	19.391486	
14	500	0.60851449	15.000000	19.391486	
15	500	0.60851449	8.000000	19.391486	
16	500	0.60851449	18.000000	19.391486	
17	500	0.60851449	12.000000	19.391486	
:	:	:	:	:	
29	500	0.60851449	7.000000	19.391486	
30	500	0.60851449	9.000000	19.391486	

Test Descriptions
Test 6   Four out of five points in a row in Zone B or beyond





Test Descriptions	
Test 6	Four out of five points in a row in Zone B or beyond

7.1.3 p-chart for Unknown p

- Quite often, the value of p will not be known. An estimate of the unknown p, which will be denoted $\overline{p}$, must be computed from preliminary data.
- Suppose m preliminary samples (usually between 20 and 25) of size n are drawn from a production process that is assumed to be in control.
- For each of the samples, $\widehat{p}_i$, for i=1,2,...,m, is computed, where $\widehat{p}_i$ is the sample fraction nonconforming from the i^{th} preliminary sample.
- The estimate of p used in constructing control limits is $\overline{p} = \dfrac{\sum_{i=1}^{m} \widehat{p}_i}{m} = \dfrac{\sum_{i=1}^{m} D_i}{mn}$.
- The **trial control limits** are computed by substituting $\overline{p}$ for p in formula (13):

$$\begin{aligned} \text{UCL} &= \overline{p} + 3 \\ \text{Centerline} &= \overline{p} \\ \text{LCL} &= \overline{p} - 3 \end{aligned} \qquad (14)$$

- The trial limits must be used to test whether or not the process was in control when the preliminary samples were taken.
- Use some subset of the rules proposed earlier to determine if any preliminary data points indicate an out-of-control process.
  - If this test indicates no out of control signals, accept the trial control limits as valid control limits for future process control testing.
  - If any points indicate an out-of-control process, an investigation for assignable causes should be carried out.
  - If an assignable cause can be found, delete the point and recompute the trial control limits. If no assignable cause can be found, one of two things can be done:
    1. The point can be deleted and new limits computed. Continue with the preceding test until acceptable limits are found.
    2. The point can be retained, along with the trial control limits. Future points can be plotted to see if they plot in control. If so, accept the limits as valid.
- Once valid control limits have been computed, testing for an in-control process can proceed.
  - Samples should be collected from the same process.
  - Compute the value of $\widehat{p}$ for each sample as the data becomes available.
  - Plot the most current value of $\widehat{p}$ on the control chart and use a subset of the rules discussed earlier to determine if the process is running in control.
- Occasionally, multiple sample $\widehat{p}_i$ values are outside of the control limits. Searching for assignable causes for each case is often a waste of time. In such cases, it is better to look for patterns formed by these values. Often the out-of-control pattern can be attributed to an assignable cause.

7.1.4 np-Chart for Unknown p

- As an alternative to the p-chart when n is constant and p is unknown, it is possible to use the corresponding np-chart.
- If p is unknown or not specified, the $\overline{p}$ can be used to estimate p.
- The **control limits for the np-chart** with estimated p are:

$$\begin{aligned} \text{UCL} &= n\overline{p} + 3 \\ \text{Centerline} &= n\overline{p} \\ \text{LCL} &= n\overline{p} - 3 \end{aligned} \tag{15}$$

SAS Code for p-chart (p unknown)

```
DM 'LOG; CLEAR; OUT; CLEAR;';
* ODS PRINTER PDF file='C:\COURSES\ST528\SAS\PCHART1.PDF';
ODS LISTING;
OPTIONS NODATE NONUMBER LS=76 PS=54;

*******************************************************;
***                  P-chart                       ***;
*** The response is the number of failing circuits ***;
***          from 20 batches of size 500           ***;
*******************************************************;
DATA circuit3;
   INPUT batch fail @@;
LINES;
 1 12    2 21    3 16    4  9    5  3    6  4    7  6    8  9
 9 11   10 13   11 12   12  7   13  2   14 14   15  9   16  8
17 14   18 10   19 11   20  9
;
TITLE 'p Chart for the Proportion of Failing Circuits';
SYMBOL1 VALUE=dot WIDTH=3;

PROC SHEWHART DATA=circuit3;
     PCHART fail*batch='1' / TESTS = 1 to 8  SUBGROUPN = 500
                       ZONELABELS TABLETESTS TABLELEGEND;
   LABEL batch = 'Batch Number'
         fail  = 'Fraction Failing';
RUN;
```

SAS Code for np-chart (p unknown)

Once again, just replace PCHART and PSYMBOL=p0 with NPCHART and NPSYMBOL = np0.

SAS Output for p-chart (p unknown)

p Chart for the Proportion of Failing Circuits

The SHEWHART Procedure

p Chart Summary for fail

batch	Subgroup Sample Size	-3 Sigma Limits with n=500 for Proportion- Lower Limit	Subgroup Proportion	Upper Limit	Special Tests Signaled
1	500	0.00121703	0.02400000	0.03878297	
2	500	0.00121703	0.04200000	0.03878297	1
3	500	0.00121703	0.03200000	0.03878297	
4	500	0.00121703	0.01800000	0.03878297	
5	500	0.00121703	0.00600000	0.03878297	
6	500	0.00121703	0.00800000	0.03878297	
7	500	0.00121703	0.01200000	0.03878297	
8	500	0.00121703	0.01800000	0.03878297	
9	500	0.00121703	0.02200000	0.03878297	
10	500	0.00121703	0.02600000	0.03878297	3
11	500	0.00121703	0.02400000	0.03878297	
12	500	0.00121703	0.01400000	0.03878297	
13	500	0.00121703	0.00400000	0.03878297	
14	500	0.00121703	0.02800000	0.03878297	
15	500	0.00121703	0.01800000	0.03878297	
16	500	0.00121703	0.01600000	0.03878297	
17	500	0.00121703	0.02800000	0.03878297	
18	500	0.00121703	0.02000000	0.03878297	
19	500	0.00121703	0.02200000	0.03878297	
20	500	0.00121703	0.01800000	0.03878297	

Test Descriptions

Test 1 One point beyond Zone A (outside control limits)
Test 3 Six points in a row steadily increasing or decreasing

SAS Output for np-chart (p unknown)

np Chart for the Proportion of Failing Circuits

The SHEWHART Procedure

np Chart Summary for fail

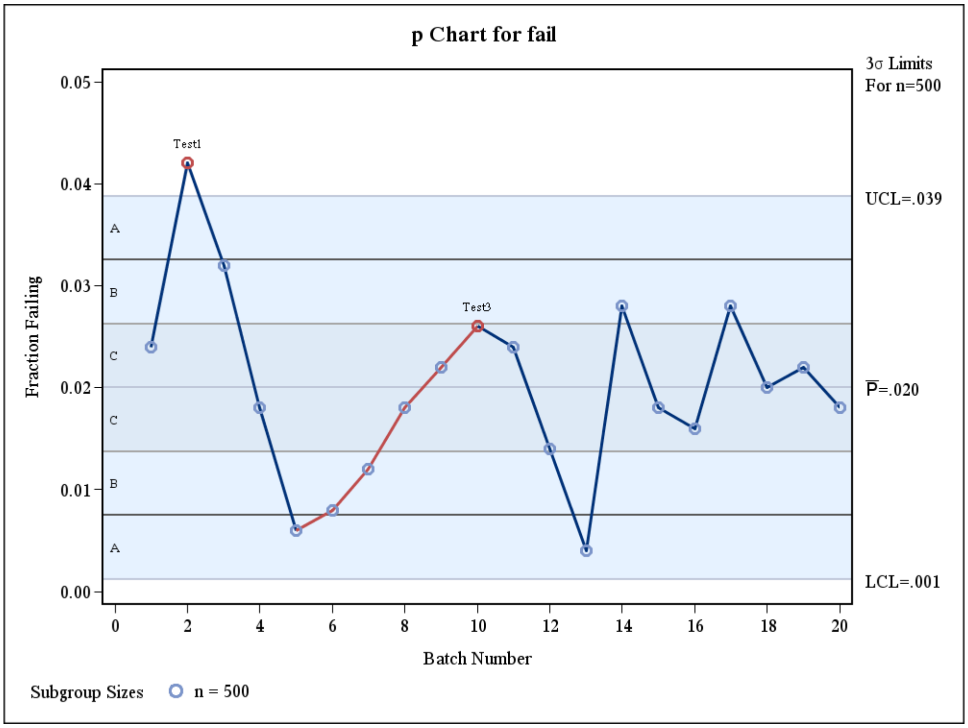
batch	Subgroup Sample Size	-3 Sigma Limits with n=500 for Number- Lower Limit	Subgroup Number	Upper Limit	Special Tests Signaled
1	500	0.60851449	12.000000	19.391486	
2	500	0.60851449	21.000000	19.391486	1
3	500	0.60851449	16.000000	19.391486	
4	500	0.60851449	9.000000	19.391486	
5	500	0.60851449	3.000000	19.391486	
6	500	0.60851449	4.000000	19.391486	
7	500	0.60851449	6.000000	19.391486	
8	500	0.60851449	9.000000	19.391486	
9	500	0.60851449	11.000000	19.391486	
10	500	0.60851449	13.000000	19.391486	3
11	500	0.60851449	12.000000	19.391486	
12	500	0.60851449	7.000000	19.391486	
13	500	0.60851449	2.000000	19.391486	
14	500	0.60851449	14.000000	19.391486	
15	500	0.60851449	9.000000	19.391486	
16	500	0.60851449	8.000000	19.391486	
17	500	0.60851449	14.000000	19.391486	
18	500	0.60851449	10.000000	19.391486	
19	500	0.60851449	11.000000	19.391486	
20	500	0.60851449	9.000000	19.391486	

Test Descriptions

Test 1 One point beyond Zone A (outside control limits)
Test 3 Six points in a row steadily increasing or decreasing

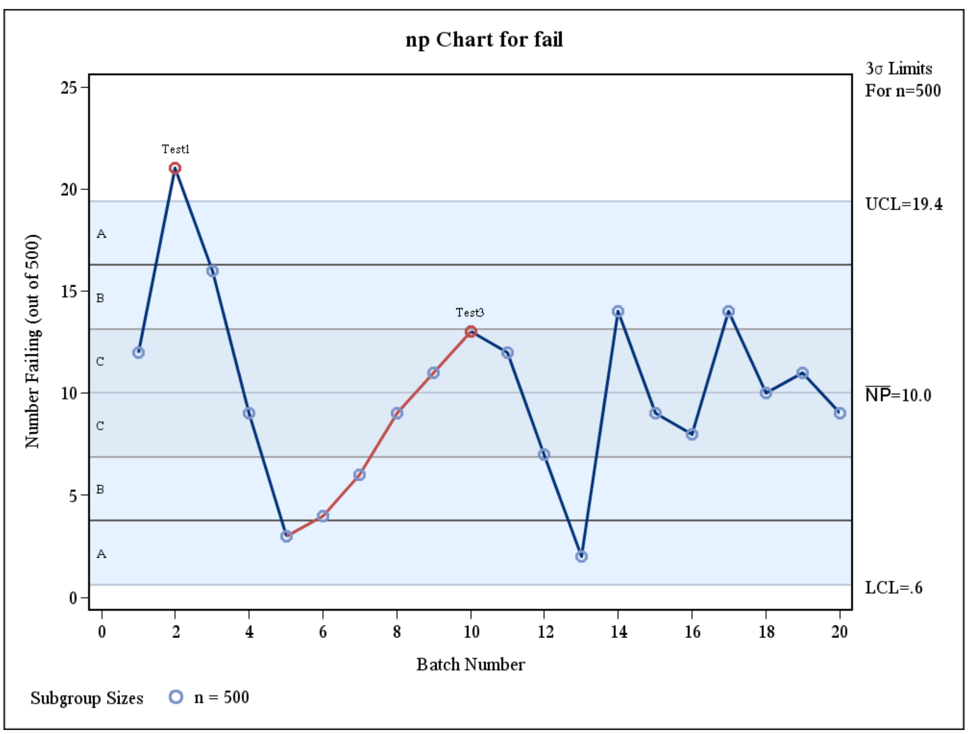
p Chart for the Proportion of Failing Circuits

The SHEWHART Procedure



np Chart for the Proportion of Failing Circuits

The SHEWHART Procedure



Test Descriptions	
Test 1	One point beyond Zone A (outside control limits)
Test 3	Six points in a row steadily increasing or decreasing

7.1.5 Design of the Fraction Nonconforming Control Chart

Three parameters must be specified: the sample size, the sampling frequency, and the width of the control limits. Various guidelines for parameter selection:

1. Select sampling frequency appropriate for the production rate yielding a fixed sample size.
2. Rational subgrouping to determine sampling frequency.
3. Economic restrictions.
4. Determining the sample size n: If p is small, choose n sufficiently large so that the probability of finding at least one nonconforming unit is high. Several possible criteria:

   (a) For any $p > 0$ there is a positive probability of producing some defectives. Therefore, it is unreasonable to conclude that the process is out of control upon observing a single nonconforming item. To avoid this situation, n is chosen so that the probability of finding at least one nonconforming unit per sample is at least some specifed value γ.

   (b) Choose n large enough so that the control chart will have a positive lower control limit: $\text{LCL} = p - L\sqrt{\frac{p(1-p)}{n}} > 0$ yielding $n > \frac{1-p}{p} L^2$. This forces an investigation when one or more samples contain an unusually small number of nonconforming items.

   (c) (Duncan 1986) Choose n large enough so that we have a 50% chance of detecting a process shift of a specified magnitude δ. Therefore, n satisfies $\delta = L\sqrt{\frac{p(1-p)}{n}}$ yielding $n = \left(\frac{L}{\delta}\right)^2 p(1-p)$. Note: $L = 3$ for standard 3σ control limits.

Interpretation of Points on the Control Chart

- Exercise caution when interpreting points at or below the lower control limit (LCL) because they often do not represent real improvement in quality. They can represent:
  - Errors in the inspection process (inadequately trained or inexperienced inspectors, uncalibrated inspection equipment).
  - Inspectors deliberately pass nonconforming items (flinching) or report fictitious data.
- The analyst must look for assignable causes. Not all downward shifts in p are attributable to improved quality.

7.2 Variable Sample Size p-chart and np-chart

- Occasionally the number of units produced will vary in each sampling period generating a control chart with variable sample size. Methods of determining trial control limits for the p-chart include:

  1. Calculate individual control limits for each sample. If n_i is the sample size of sample i, then UCL and LCL are $\overline{p} \pm 3$ where $\overline{p} = \frac{\sum_{i=1}^{m} D_i}{\sum_{i=1}^{m} n_i}$.
  2. Base the control chart on the average sample size $\overline{n}$. This provides approximate (but constant) control limits $\overline{p} \pm 3$
     - Assumption: future sample sizes will be similar to those already observed.
     - Warning: if there is an unusually large variation in the size of a particular sample or if a point is close to the control limits, then exact control limits should be determined for these points.
  3. Use a standardized p-chart.

- For an np-chart with variable n, the UCL and LCL control limits for the np-chart with estimated p are either:

$$n_i\overline{p} \pm 3 \qquad \text{or} \qquad \overline{np} \pm 3$$

7.2.1 Standardized p-charts

- The standardized i^{th} sample proportion nonconforming is $Z_i =$

  Thus, when the process is in-control, Z_i values should be approximately $N(0,1)$.

- The Z_i values are plotted on a control chart with control limits set at ± 3.

SAS Code for p-chart (p unknown and unequal n)

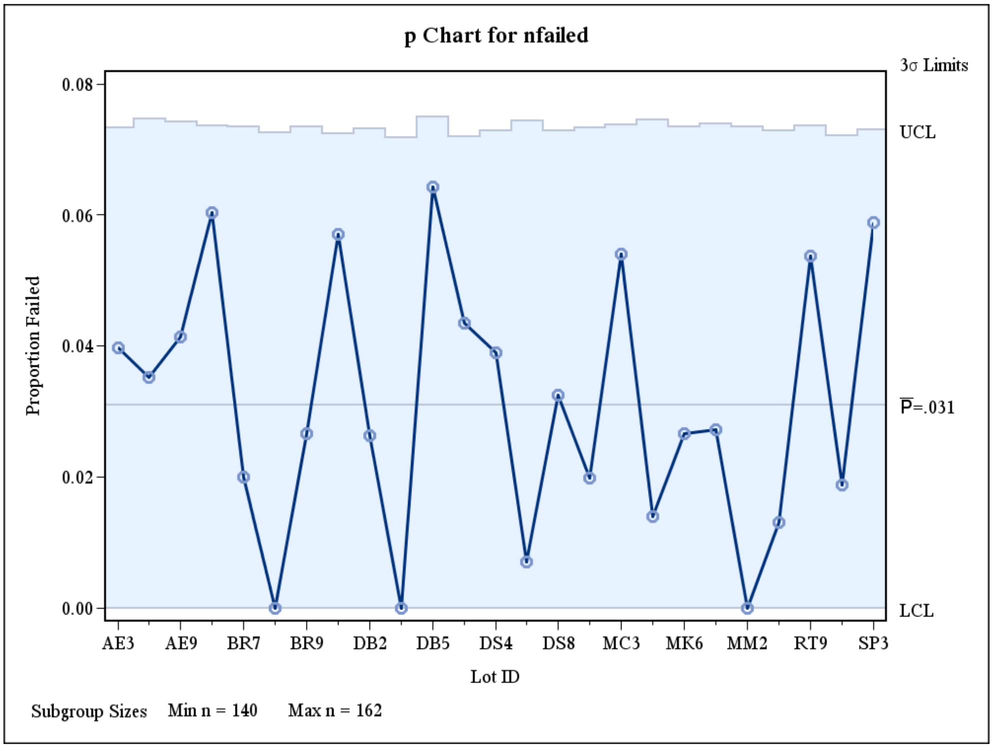
```
*************************************************************;
*** p-chart (with unequal sample sizes).  The response is the ***;
*** number of failing batteries from 25 lots of varying size  ***;
*************************************************************;
 DATA battery;
    INPUT lot $ nfailed sampsize @@;;
    LABEL nfailed ='Number Failed'
          lot     ='Lot Identifier'
          sampsize='Number Sampled';
 LINES;
 AE3 6 151   AE4 5 142   AE9 6 145   BR3 9 149   BR7 3 150   BR8 0 156
 BR9 4 150   DB1 9 158   DB2 4 152   DB3 0 162   DB5 9 140   DB6 7 161
 DS4 6 154   DS6 1 144   DS8 5 154   JG1 3 151   MC3 8 148   MC4 2 143
 MK6 4 150   MM1 4 147   MM2 0 150   RT5 2 154   RT9 8 149   SP1 3 160
 SP3 9 153
 ;
 /* The p chart has control limits that vary with the subgroup sample size. */
 TITLE 'Proportion of Battery Failures';
 SYMBOL1 VALUE=square WIDTH=3;

 PROC SHEWHART DATA=battery ;
    PCHART nfailed*lot='1' / TESTS = 1 to 8  SUBGROUPN = sampsize
                         TABLETESTS TABLELEGEND;
    LABEL nfailed='Proportion Failed'   lot = 'Lot ID';
 RUN;
```

SAS Code for np-chart (p unknown and unequal n) Change PCHART to NPCHART.

SAS Output for p-chart (p unknown and unequal n)

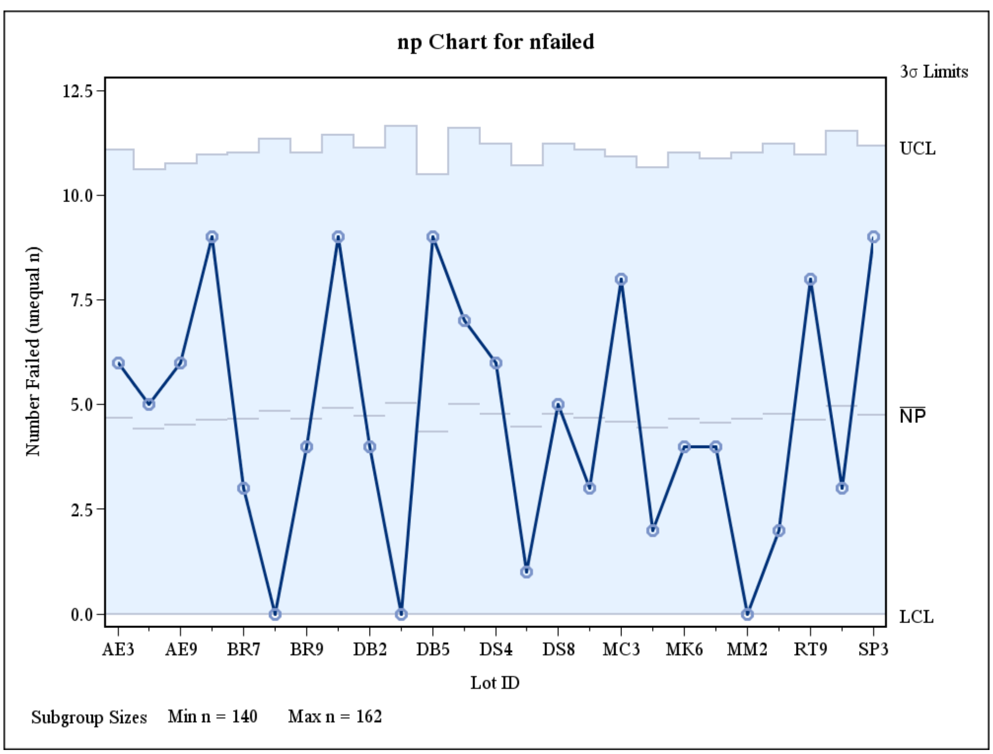
Proportion of Battery Failures

The SHEWHART Procedure



Control Limits for Battery Failures

The SHEWHART Procedure



Proportion of Battery Failures

The SHEWHART Procedure

p Chart Summary for nfailed

lot	Subgroup Sample Size	--3 Sigma Limits for Proportion-- Lower Limit	Subgroup Proportion	Upper Limit	Special Tests Signaled
AE3	151	0	0.03973510	0.07332945	
AE4	142	0	0.03521127	0.07464996	
AE9	145	0	0.04137931	0.07419615	
BR3	149	0	0.06040268	0.07361253	
BR7	150	0	0.02000000	0.07347028	
BR8	156	0	0.00000000	0.07264573	
BR9	150	0	0.02666667	0.07347028	
DB1	158	0	0.05696203	0.07238137	
DB2	152	0	0.02631579	0.07319001	
DB3	162	0	0.00000000	0.07186742	
DB5	140	0	0.06428571	0.07496057	
DB6	161	0	0.04347826	0.07199411	
DS4	154	0	0.03896104	0.07291522	
DS6	144	0	0.00694444	0.07434585	
DS8	154	0	0.03246753	0.07291522	
JG1	151	0	0.01986755	0.07332945	
MC3	148	0	0.05405405	0.07375621	
MC4	143	0	0.01398601	0.07449711	
MK6	150	0	0.02666667	0.07347028	
MM1	147	0	0.02721088	0.07390136	
MM2	150	0	0.00000000	0.07347028	
RT5	154	0	0.01298701	0.07291522	
RT9	149	0	0.05369128	0.07361253	
SP1	160	0	0.01875000	0.07212198	
SP3	153	0	0.05882353	0.07305194	

7.3 Operating Characteristic Curves

- An *Operating Characteristic (OC)* curve of the fraction nonconforming is a graphical display of the probability of incorrectly accepting the hypothesis of statistical control (β risk) against the process fraction nonconforming p.
- The OC curve provides a measure of the *sensitivity* of the control chart (its ability to detect a shift in the process fraction nonconforming from the nominal value $\bar{p}$ to some other value p).

- β_p can be determined from a binomial CDF with parameters n and p.

- Recall: $\text{ARL} = \dfrac{1}{P(\text{sample point plots out of control})}$

- If the process is in-control then $\text{ARL}_0 = \dfrac{1}{\alpha}$.

- If the process is out-of-control then $\text{ARL}_1 = \dfrac{1}{1-\beta_p}$.

See my course webpage for the **SAS Code for making an OCC for a p-chart**. I modified the code in the SAS help manual.